

Elongation of Moving Noncommutative Solitons

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We discuss the characteristic properties of noncommutative solitons moving with constant velocity. As noncommutativity breaks the Lorentz symmetry, the shape of moving solitons is affected not just by the Lorentz contraction along the velocity direction, but also sometimes by additional ‘elongation’ transverse to the velocity direction. We explore this in two examples: noncommutative solitons in a scalar field theory on two spatial dimension and ‘long stick’ shaped noncommutative $U(2)$ magnetic monopoles. However the elongation factors of these two cases are different, and so not universal.

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Solitons in the noncommutative field theories have attracted much attention recently[1, 2, 3, 4, 5, 6, 7, 8, 9]. Localized solitons in a noncommutative scalar theory of spatial dimensions higher than one is already peculiar because they lost their identity in the commutative case as dictated by the Hobart-Derrick theorem[5, 11, 12]. On the other hand, monopoles or dyons of the commutative super Yang-Mills (SYM) theories become a sticklike. From the D-brane picture, D-strings connecting D3 branes become tilted in noncommutative case. In the field theory picture, the image of a D-string on three space appears as a finite segment of Dirac string, whose two ends are like Dirac magnetic monopoles of two different $U(1)$ subgroups of $U(2)$ [2, 3, 4, 7, 8, 9].

The detailed dynamical aspects of these noncommutative solitons are not much pursued. Investigation of the free motion of one noncommutative soliton will be the first step toward the understanding of their solitonic moduli dynamics. (See Ref. [10] for the the moduli space of a single caloron made of N monopoles in $U(N)$ theory on noncommutative $R^3 \times S^1$.)

In this note, we consider free motions of a noncommutative soliton, which are not trivial because the systems lack the Lorentz invariance. The change of the shape of the soliton, for example, is not just dictated by the Lorentz contraction but further deformation is induced since the effective noncommutativity scale is changed due to the structure of $*$ -product. The key finding is that these solitons can be elongated along transverse to the velocity direction, but the elongation effect is not universal.

For the solitons of two dimensional scalar theory, the size transverse to the motion is elongated by the factor $\sqrt{\gamma}$ while the longitudinal size becomes contracted by the factor $1/\sqrt{\gamma}$, preserving the area size of the soliton, where the Lorentz contraction factor is $\gamma^{-1} = \sqrt{1 - v^2}$. In particular, when the velocity approaches the light velocity, the noncommutative soliton looks like a very long and thin string stretched in the transverse direction of the motion.

In the case of $U(2)$ BPS monopoles interpreted as tilted D-strings connecting two parallel $D3$ branes, the tilting is affected by the motion. Equally, the length of the Dirac string connecting two different $U(1)$ monopoles is affected. When the direction of motion is transverse to the Dirac string, the string length gets elongated by the factor γ . When the direction of motion is parallel to the string, the length is contracted by the factor γ^{-1} . As a by product, we get tilting and the tension of static (p_e, q_m) - dyons in similar perspective.

Noncommutative solitons in (2+1) dimensional scalar field theory

We shall first consider the noncommutative soliton arising in 2+1 dimensional scalar theory,

$$L = \int d^2x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right), \quad (1)$$

where the product between fields is defined by the $*$ -product,

$$a(\mathbf{x}) * b(\mathbf{x}) \equiv \left(e^{\frac{i}{2} \theta \epsilon^{ij} \partial_i \partial'_j} a(\mathbf{x}) b(\mathbf{x}') \right) \Big|_{\mathbf{x}=\mathbf{x}'}. \quad (2)$$

For simplicity, we shall consider the case where the potential has its absolute minimum at $V(0) = 0$ and the other local minimum at some λ with $V(\lambda) > 0$. As shown in detail in Ref. [5], the localized soliton corresponds to a false vacuum bubble where the noncommutativity prevents its collapse to a zero size. This is contrasted to the case of commutative scalar field theory where localized solitons do not exist at all as dictated by the Hobart-Derrick theorem[11, 12].

The static localized noncommutative soliton solution will satisfy

$$-\nabla^2\phi + V'(\phi) = 0. \quad (3)$$

In the large θ limit, the potential part is dominant over the kinetic contributions. This can be easily shown by introducing dimensionless coordinates $\tilde{x}_i = x_i/\sqrt{\theta}$. The energy functional is written as

$$E = \int d^2\tilde{x} \left(\frac{1}{2} |\tilde{\nabla}\phi|^2 + \theta V(\phi) \right). \quad (4)$$

where the $*$ -product is defined in terms of \tilde{x} with $\theta = 1$. Thus we clearly see that the kinetic contribution is negligible compared to the potential contribution. Neglecting the kinetic term, the static normalizable solution in this limit can be constructed with help of projection function,

$$P_n(\mathbf{r}) = 2(-1)^n e^{-\frac{r^2}{\theta}} L_n(2r^2/\theta) \quad (5)$$

where $L_n(x)$ is the n -th order Laguerre polynomial. These functions work as projection operators under the $*$ -product; they satisfy $P_n * P_m = \delta_{nm} P_n$ [13]. The most general radially symmetric normalizable solutions of $V'(\phi) = 0$ are then given by

$$\phi = \sum_n w_n P_n, \quad (6)$$

where w_n belongs to the set $\{\lambda_l\}$ of real extrema of $V(x)$. Let us, for example, take the simplest solution,

$$\phi(x, y) = \lambda P_0(x, y) = 2\lambda e^{-r^2/\theta}. \quad (7)$$

The size of the soliton is approximately $R = \sqrt{\theta}$. By the axial symmetry, x-directional size $L_x = \sqrt{\theta}$ is the same as the y-directional size $L_y = \sqrt{\theta}$. Let us call their potential and kinetic energy to be

$$K_0 = \int d^2\tilde{x} \frac{1}{2} (\tilde{\nabla}\phi)^2, \quad U_0 = \theta \int d^2\tilde{x} V(\phi) \quad (8)$$

Their order of magnitude is $K_0 \sim \mathcal{O}(1)$ and $U_0 \sim \mathcal{O}(\theta)$. Thus the rest mass, $E_0 = K_0 + U_0$, is dominated by the potential.

Now let us consider any static solution $\bar{\phi}(x, y; \tilde{\theta})$ of (3) with a noncommutativity scale $\tilde{\theta}$. Due to the rotational symmetry, we consider just a soliton moving along the x axis. In the commutative case, the time dependent solution describing a moving profile can be constructed by boosting the static soliton by

$$t' = \gamma(t - vx), \quad x' = \gamma(x - vt), \quad y' = y, \quad (9)$$

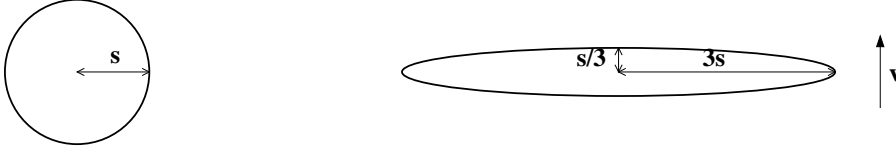


Figure 1: The static soliton with $s \equiv \sqrt{\theta}$ and the shape of moving soliton with $\gamma = 9$.

with $\gamma = 1/\sqrt{1-v^2}$. Because the symmetry under the Lorentz boost is explicitly broken by the noncommutativity, the Lorentz boost no longer generates a new solution. Instead, the moving solution is given by

$$\phi_v(x, y, t; \theta) = \bar{\phi}(\gamma(x - vt), y; \gamma\theta), \quad (10)$$

which satisfies the equation of motion,

$$\partial_t^2 \phi_v - \nabla^2 \phi_v + V'(\phi_v) = 0. \quad (11)$$

Namely the solution is obtained not by a simple Lorentz boost but by the boost accompanied by rescaling of θ .

The deformation of the shape is not just a conventional Lorentz contraction because the effective noncommutative scale θ_{eff} is now $\gamma\theta$. For a given instant of time t , the size in each direction is now

$$L'_x(v) = \frac{\sqrt{\theta}}{\sqrt{\gamma}} = \frac{L_x}{\sqrt{\gamma}}, \quad L'_y(v) = \sqrt{\gamma} \sqrt{\theta} = \sqrt{\gamma} L_y. \quad (12)$$

It is interesting to note that the area size of the noncommutative soliton is preserved as

$$A(v) = L'_x(v) L'_y(v) = L_x L_y. \quad (13)$$

This area preserving character is consistent with the fact that the uncertainty relation set by the noncommutativity of the coordinate $\Delta x \Delta y \sim \theta$. As argued in Ref. [5], this uncertainty relation is responsible for the size of the soliton, without which the soliton would collapse to zero size. As v grows, the transverse size to the motion grows as $\sqrt{\gamma}$, reflecting another UV/IR mixing of the noncommutative field theory. However the growth differs from those observed in the wave function of quantum bound state[14] or in the dipole nature described in Ref. [15]. The velocity dependence of the size is illustrated in Fig. 1.

For large θ , the kinetic energy contribution to the energy is still of $O(1)$ in θ as

$$K(v) = \frac{1}{2} \int d^2x (|\dot{\phi}|^2 + |\nabla \phi|^2) = \frac{1}{2\gamma} \int d^2x' (\gamma^2(1+v^2)|\partial_{x'} \phi|^2 + |\partial_{y'} \phi|^2) = \gamma K_0 \quad (14)$$

where we have used the fact that the soliton at the rest frame is rotationally symmetric. Thus the potential energy contribution is dominant over the kinetic part and, consequently, the potential energy is given by

$$U(v) = \int d^2x V(\phi(\gamma(x - vt), y; \gamma\theta)) = \frac{1}{\gamma} \int d^2x' V(\phi(x', y'; \theta\gamma)) = U_0. \quad (15)$$

Here we have used the fact that U_0 is linear in θ . Thus the total energy transforms as

$$E(v) = \frac{1}{\sqrt{1-v^2}} K_0 + U_0. \quad (16)$$

In the large θ limit, the potential energy is dominant until the velocity is highly relativistic so that $v \sim \sqrt{1 - (K_0/U_0)^2}$. In case of ordinary solitons in 1+1 dimensional sine-Gordon model or monopoles in SYM theories, the energy scales as those of ordinary massive particles; $E(v) = \gamma E_0$. Thus, the behavior of energy of the noncommutative solitons is again quite different from that of the conventional soliton. We now turn to the case of momentum of the moving soliton. Using the translational invariance of the system, the conserved momentum may be constructed using the Noether procedure and the resulting expression reads,

$$\mathbf{P} = \int d^2x \partial_t \phi \nabla \phi. \quad (17)$$

The momentum of the noncommutative soliton is then evaluated as

$$P_x = \gamma v \int d^2x' \left(\frac{\partial \phi}{\partial x'} \right)^2 = v \gamma K_0 \quad (18)$$

Hence the momentum is not given by $\gamma v E_0$ but its value is much smaller compared to a particle with rest mass E_0 .

Because of the change of the shape of moving noncommutative solitons, the characteristic of classical scattering, for example, ought to differ from ordinary particles with short-ranged interactions. As the relative velocity grows, the size felt becomes bigger, and the cross section is expected to grow, though a detailed analysis is necessary to see this effect explicitly.

Noncommutative $U(2)$ monopole and (p_e, q_m) -dyons

We begin by recapitulating the static properties of noncommutative monopole in the N=4 supersymmetric Yang-Mills theory. We shall restrict our discussion to the case of $U(2)$ gauge group. Among the six Higgs fields, only a Higgs field ϕ plays a role in the following discussion. The bosonic part of the action is then given by

$$S = \frac{1}{g_{\text{YM}}^2} \int d^4x \text{tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi D^\mu \phi \right). \quad (19)$$

We shall take the only nonvanishing components to be $\theta_{12} = -\theta_{21} \equiv \theta$. The four vector potential and ϕ belong to $U(2)$ Lie algebra generated by $\frac{1}{2} I_{2 \times 2}$ and $\frac{1}{2} (\sigma_1, \sigma_2, \sigma_3)$. We set the vacuum expectation value of the Higgs field in the asymptotic region as $u \sigma_3 / 2$.

The energy functional

$$E = \frac{1}{2g_{\text{YM}}^2} \int d^3x \text{tr} \left(E^2 + |D_0 \phi|^2 + B^2 + |D\phi|^2 \right) \geq \frac{1}{g_{\text{YM}}^2} \int_{r=\infty} dS_k \text{tr} B_k \phi, \quad (20)$$

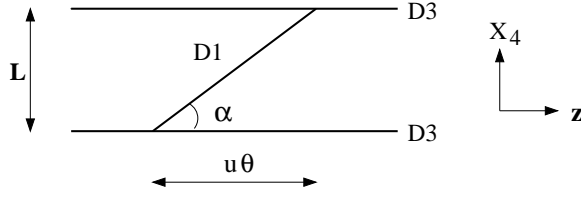


Figure 2: The brane configuration of a noncommutative monopole where $L = u l_s^2$ and $\cot \alpha = u\theta/L$.

is bounded as in the case of the ordinary supersymmetric Yang-Mills theory. The bound is saturated if the BPS equation

$$B = D\phi. \quad (21)$$

The mass for the monopole solution is then

$$M = \frac{2\pi q_m}{g_{\text{YM}}^2} u \quad (22)$$

where we define the magnetic charge Q_M by

$$q_m = \frac{1}{2\pi u} \int_{r=\infty} dS_k \text{tr} B_k \phi. \quad (23)$$

The charge is to be quantized at integer values even in the noncommutative case. This is because the fields in the asymptotic region are slowly varying and, hence, the standard argument of the topological quantization of the magnetic charge holds.

The $U(2)$ noncommutativity monopole solution has been investigated in Ref. [2, 3, 4, 8] and the solution to the second order in θ has been found. In Ref. [8] the full brane configuration in the commutative SYM picture was found, which is related to the noncommutative description via the Seiberg-Witten map[16]. The monopole (D-string) is tilted between two parallel D3 branes as schematically illustrated in Fig. 2, where we interpret ϕl_s^2 as the transverse coordinate X_4 . The configuration has an axial symmetry along z-axis and the projected image to the three space has the z-directional size $L_z = u\theta$. The gauge symmetry $U(2)$ is spontaneously broken to $U(1) \times U(1)'$. The two points where D string meet two D3 branes are like $U(1)$ monopole and $U(1)'$ anti-monopole, which are now separated in finite interval of length $u\theta$ along the z direction and connected by the Dirac string of finite tension. This Dirac string is the image of the tilted D-string on three space.

This size L_z measures the extension of this Dirac string along z-direction. The distance between two D3-branes is given by $L = u l_s^2$ with the string length scale $l_s = \sqrt{2\pi\alpha'}$. Consequently the tilting angle α is

$$\alpha = \tan^{-1} l_s^2 / \theta, \quad (24)$$

which does to zero in the zero slope limit. Since the length of monopole or D-string is then $\sqrt{L^2 + L_z^2}$, the tension of the monopole is

$$T_0 = \frac{M}{\sqrt{L^2 + (u\theta)^2}} = \frac{2\pi}{g_{\text{YM}}^2} \frac{q_m}{\sqrt{l_s^4 + \theta^2}}. \quad (25)$$

In particular, in the zero slope limit of $2\pi\alpha' \rightarrow 0$, the tension becomes identical to the tension of the Dirac string $M/(u\theta) = \frac{2\pi q_m}{g_{\text{YM}}^2 \theta}$, which agrees with the value in [9]. (In the full field theory, this should be true when u is very large and so the Dirac string is very long. Then the field energy of two $U(1)$ monopoles can be negligible in comparison.)

The solution of a moving monopole can be generated from the static solution. Let $\bar{A}_\mu(x, y, z; \theta)$ and $\bar{\phi}(x, y, z; \theta)$ be the solution describing a static monopole for arbitrary θ . For monopole moving on the $x - y$ plane, we restrict our discussion to $\mathbf{v} = v\hat{x}$ due to the rotation symmetry around the z axis. The corresponding Lorentz transformation is given by

$$t' = \gamma(t - vx), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z. \quad (26)$$

It is now straightforward to verify that the the solution of a moving monopole is

$$\begin{aligned} A'_\mu(t, x, y, z; \theta) &= \frac{\partial x'^\nu}{\partial x^\mu} \bar{A}_\nu(\gamma(x - vt), y, z; \gamma\theta) \\ \phi'(t, x, y, z; \theta) &= \bar{\phi}(\gamma(x - vt), y, z; \gamma\theta). \end{aligned} \quad (27)$$

The effective size in the z direction is given by $L_z(v) = u\theta\gamma = \gamma L_z$. Since the effective noncommutative scale θ_{eff} is given by $\gamma\theta$, the monopole looks more tilted when it is moving on $x - y$ plane. Namely, the distance $L = ul_s^2$ between D3-branes unchanged but the Dirac string connecting $U(1)$ monopole and $U(1)'$ anti-monopole gets ‘elongated’ by the factor γ , and so the size becomes $L_z(v) = \gamma u\theta$. The tilted slope angle is now given by $\alpha(v) = \tan^{-1}(l_s^2/\gamma\theta)$. Thus, in the relativistic speed, the $U(2)$ monopole would look like a very long stick. A few comments are in order. First, it should be noted that the full $U(2)$ solution is not known thus far for finite θ . Hence we do not have the explicit solution for the finite velocity either. Secondly, the shape of monopole is governed by the noncommutativity scale θ together with the scale L_z that controls the dipole structure. The whole deformation of the monopole shape is partly from the change of the dipole structure as the scale L_z changes effectively. There is also deformation from the Lorentz contraction and the change in the effective noncommutativity scale. The latter part of deformation is universal to all moving noncommutative solitons including the case of scalar noncommutative solitons.

The moving solution to the z direction is obtained similarly. The Lorentz boost transformation in the z direction reads

$$t' = \gamma(t - vz), \quad x' = x, \quad y' = y, \quad z' = \gamma(z - vt). \quad (28)$$

The the corresponding moving solution is given by

$$\begin{aligned} A'_\mu(t, x, y, z; \theta) &= \frac{\partial x'^\nu}{\partial x^\mu} \bar{A}_\nu(x, y, \gamma(z - vt); \theta) \\ \phi'(t, x, y, z; \theta) &= \bar{\phi}(x, y, \gamma(z - vt); \theta), \end{aligned} \quad (29)$$

and the effective noncommutative scale remains unchanged and only ordinary Lorentz contraction in the z direction by the factor $1/\gamma$ has occurred. The image of the $U(2)$ monopole in the three space has the size $L_z(v) = u\theta/\gamma$ and titling angle becomes $\alpha(v) = \tan^{-1}(\gamma l_s^2/\theta)$.

Contrary to the case of the scalar noncommutative soliton, the energy and momentum of the moving monopole behaves like a massive particle. Namely, they are respectively given by $E = \gamma M$ and $\mathbf{P} = \gamma M \mathbf{v}$, which may be checked directly by inserting the above solutions to definitions of the energy and the momentum.

Finally let us consider the case of dyons or (p_e, q_m) -strings. The dyons satisfy BPS equations

$$\begin{aligned} B &= \cos \xi D\phi \\ E &= \sin \xi D\phi. \end{aligned} \quad (30)$$

We define electric charge by

$$p_e = \frac{1}{g_{YM}^2 u} \int_{r=\infty} dS_k \text{tr } E_k \phi, \quad (31)$$

and, for the elementary excitations of W-bosons, it takes integer values as expected. Since E and B are related through the angle ξ in the above BPS equations, the ratio of the electric charge p_e and the magnetic charge q_m is then found to be

$$\frac{p_e}{q_m} = \frac{2\pi}{g_{YM}^2} \tan \xi. \quad (32)$$

For a given magnetic solution, the corresponding dyon solution can be found by a scale transformation and a Lorentz boost in the extra dimension if we view as $\phi = A_4$. In the noncommutative case, one should take into account the change of the effective noncommutative scale. The corresponding dyon solution satisfying the above BPS equations are

$$\begin{aligned} A_i &= \bar{A}_i(\mathbf{r} \cos \xi; \theta \cos^2 \xi) \\ \phi &= \bar{\phi}(\mathbf{r} \cos \xi; \theta \cos^2 \xi) \\ A_0 &= \sin \xi \bar{\phi}(\mathbf{r} \cos \xi; \theta \cos^2 \xi) \end{aligned} \quad (33)$$

The energy is determined again by surface integral and can be given in terms of charges by

$$M_{p_e, q_m} = \frac{2\pi}{g_{YM}^2} \frac{q_m u}{\cos \xi} = u \sqrt{\left(\frac{2\pi q_m}{g_{YM}^2} \right)^2 + p_e^2}. \quad (34)$$

The length scale of the image in z direction is shrunken to $L_z^D = u\theta / \sqrt{1 + (g_{YM}^2 p_e / 2\pi q_m)^2}$ and the tilting angle changes to $\alpha_D = \tan^{-1}((l_s^2 / \theta) \sqrt{1 + (g_{YM}^2 p_e / 2\pi q_m)^2})$. In the zero slope limit, the string tension for the Dirac string becomes

$$T_{(p_e, q_m)} = \frac{2\pi q_m}{g_{YM}^2 \theta} \left(1 + (g_{YM}^2 p_e / 2\pi q_m)^2 \right), \quad (35)$$

for $q_m \neq 0$. When $p_e = 0$, the tension becomes $T = \frac{2\pi q}{g_{YM}^2 \theta}$, which agrees with the value given previously. When $q_m = 0$, Eq. (35) is not valid. Indeed the fundamental string tension goes

to infinite in the zero slop limit. When moving, these (p_e, q_m) dyonic configurations would also go through the same elongation or contraction as the pure monopole configuration.

In this note, we have observed the shape of moving noncommutative solitons is elongated. In short, it accentuates the UV/IR mixing. As the velocity approaches the light velocity, the transverse size grows indefinitely, which is a phenomena residing in the IR regime of the theory. Moving monopoles in the noncommutative SYM theories can have a similar elongation, following the change of tilting of D-strings connecting D3 branes. We have obtained the tension and tilting of static (p_e, q_m) -dyons.

Investigation of the free motion is the first step toward understanding of dynamical characteristics of noncommutative solitons. For the more detailed dynamics, further studies are required on the moduli dynamics of noncommutative solitons. Especially, the quantum moduli dynamics of the false vacuum bubble in the noncommutative scalar field theory will be of interest. Also our observation on monopoles would also apply to the noncommutative open string theories studied recently [17].

Acknowledgment We would like to thank P. Yi and C. Zachos for enlightening discussions and R. Jackiw for pointing out Ref. [11]. D.B. would like to thank H. S. Yang for valuable comments. This work is supported in part by KOSEF 1998 Interdisciplinary Research Grant 98-07-02-07-01-5 (DB and KL) and by UOS Academic Research Grant (DB). K.L. thanks the hospitality in Niels Bohr Institute where the part of work is done.

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